Chapter 6.1 part 1

Chapter 6 Ideals and quotient rings

Let R, S be rings, and let $f: R \to S$ be a ring homomorphism

$f(a+b) = f(a) + f(b)$ | A map which respects
$f(ab) = f(a) f(b)$ | the ring structure

Recall
Cor 3.11 If $f: R \to S$ is a ring homomorphism, then the image of is a subring in S.

$\underline{\mathrm{Im}\, f} = \{ f(a) \mid a \in R \} \subseteq S$ | The map f is surjective iff $\underline{\mathrm{Im}\, f = S}$

Def kernel of a homomorphism $f: R \to S$

$\underline{\mathrm{Ker}\, f} = \{ a \in R \mid f(a) = 0_S \} \subseteq R \neq \emptyset$ $\qquad f(0_R) = 0_S$

Th 6.11 The map f is injective iff $\underline{\mathrm{Ker}\, f = \{0_R\}}$

<u>Prop</u> For a ring homomorphism $f: R \to S$, $\mathrm{Ker}\, f$ is a <u>subring</u> in R.

<u>Pf</u>

Criterion to check
Th 3.6 A non-empty subset $K \subseteq R$ is a subring iff
(1) $a, b \in K$ implies $a - b \in K$
(2) $a, b \in K$ implies $ab \in K$

(1) $a, b \in \mathrm{Ker}\, f$ means $f(a) = 0_S$ and $f(b) = 0_S$
Thus $f(a-b) = f(a) - f(b) = 0_S - 0_S = 0_S$
means $a - b \in \mathrm{Ker}\, f$ ✓

(2) $a, b \in \mathrm{Ker}\, f$ means $f(a) = 0_S$ and $f(b) = 0_S$
Thus $f(ab) = f(a) \cdot f(b) = 0_S \cdot 0_S = 0_S$ means $ab \in \mathrm{Ker}\, f$ ✓

Overkill!

We can strengthen (2):

$r \in R$ $a \in \operatorname{Ker} f$ $f(ar) = f(a)f(r) = 0_S f(r) = 0_S$ means $ar \in \operatorname{Ker} f$

$f(ra) = f(r)f(a) = f(r)0_S = 0_S$ means $ra \in \operatorname{Ker} f$

Def A subring I of a ring R is called an <u>ideal</u> (in R) provided:

whenever $r \in R$ and $a \in I$, then $ar \in I$ and $ra \in I$ } absorbtion property

Proved:

Th 6.10 Let $f: R \to S$ be a homomorphism of rings

Then $\operatorname{Ker} f$ is an ideal in R

Ex 6 Not every subring is an ideal $\mathbb{Z} \subset \mathbb{Q}$ - does not have the absorbtion property

Ideals are very special subrings.

Th 6.1 A non-empty subset $I \subset R$ of a ring R is an ideal iff

(1) $a, b \in I$ implies $a - b \in I$

(2) $a \in I, r \in R$ implies $ar \in I$ and $ra \in I$

Examples $R \supset \{0_R\}$ - ideal
$R \supseteq R$ - ideal
} Trivial

$\mathbb{Z}$, $n \in \mathbb{Z}$ $\mathbb{Z} \supset \{na \mid a \in \mathbb{Z}\}$ - ideal (becomes trivial if $n=0$, $n=\pm 1$)

$F[x]$, $p \in F[x]$ $\quad F[x] \supset \{fp \mid f \in F[x]\}$ - ideal (becomes trivial if $p = 0_F$ or p is a unit in $F[x]$)

Th 6.2 Let R be a commutative ring with identity.

For $c \in R$, the set $\underbrace{I = \{rc \mid r \in R\}}_{\text{def: principal ideal}}$ is an ideal.

More generally:

Th 6.3 Let R be a commutative ring with identity.

Let $c_1, \dots, c_n$ be a collection of elements of R.

Then the set $\underbrace{I = \{r_1c_1 + \dots + r_nc_n \mid r_1, r_2, \dots, r_n \in R\}}_{\text{def: finitely generated ideal}}$ is an ideal

Ex 9.8 $\quad R = \mathbb{Z}[x] \quad c_2 = 2 \quad c_1 = x$

$I = \{xf + 2g \mid f, g \in \mathbb{Z}[x]\}$ - finitely generated ideal

Exercise 15 I - the set of all polynomials in $\mathbb{Z}[x]$ with even constant term.

Let $h = 2c_0 + c_1x + \dots + c_nx^n \quad c_0, \dots c_n \in \mathbb{Z}$. Then

$h = x \cdot f + 2g$ with $g = c_0 \in \mathbb{Z}[x]$, constant polynomial

$$f = \begin{cases} 0 & \text{if } \deg h = 0 \text{ or } h = 0 \\ c_1 + c_2x + \dots + c_nx^{n-1} \end{cases}$$

Prop I is not a principal ideal

There are rings which have non-principal ideals

Pf

Assume $I = \{pf \mid f \in \mathbb{Z}[x]\}$

Wanted: such $p \in \mathbb{Z}[x]$ cannot exist

$2 \in I$, thus $2 = pf_1$ with $f_1 \in \mathbb{Z}[x]$

$\deg(2) = \deg p + \deg f_1$

$0 = \deg p + \deg f_1$ implies $\deg p = 0$ $\quad p \in \mathbb{Z}$

$x \in I$, thus $x = pf_2$ with $f_2 \in \mathbb{Z}[x]$

$\deg x = \deg p + \deg f_2$

$1 = 0 + 1$ $\quad \deg f_2 = 1$ implies $f_2 = ax + b$, $a \neq 0$

$x = p(ax+b)$

$x = pax + b$ implies $b = 0$, $pa = 1$ implies $p = 1$ or $p = -1$.

$p, a \in \mathbb{Z}$

In either case, $I = \mathbb{Z}[x]$

That is not true because I is the set of all polynomials with even constant terms while $\mathbb{Z}[x]$ contains polynomials with odd constant terms.

Terminology

The word "ideal" - ideal numbers - Dedekind thought about any ideal as "principal" generated by an "ideal number" - an element of the ring which actually does not exist.